Twistorial and space-time descriptions of $D = 4$ string models*

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Abstract

We derive twistorial tensionful bosonic string action by considering on the world sheet the canonical twistorial 2–form in two–twistor space. We demonstrate the equivalence of or model to two known momentum formulations of $D = 4$ bosonic string, with covariant worldsheet vectorial string momenta $P_\mu^m(\tau,\sigma)$ and the one with tensorial string momenta $P_{[\mu\nu]}(\tau,\sigma)$. All considered here string actions, in twistorial and mixed spinor-spacetime formulations, are classically equivalent to the Nambu-Goto action.

1 Introduction

Twistors and supertwistors [1, 2] have been often used for the description of (super) particles and (super) strings, as an alternative to space–time approach (see *e.g.* [3]-[6]). Recently large class of perturbative amplitudes in $N = 4$ $D = 4$ supersymmetric Yang-Mills theory [7] and conformal supergravity (see e.g. [8]) were described in a simple way by using strings moving in supertwistor space.

Our main aim of this report which is based on our paper [9] is to derive the twistorial master action is classically equivalent to $D = 4$ Nambu-Goto tensionful string model.

We present here the twistorial formulation of tensionful string in contrast with other studies which provide only the twistorial null (super)string and twistorial massless (super)particle models. In this paper we shall consider pure space–time formulation, pure twistor description and intermediate mixed space-time–twistorial formulation.

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In order to relate space-time and twistor target geometries for the formulation of the classical theory we shall employ the Penrose incidence relations.

Twistorial formulation of massive particles with spin [10, 11, 12, 13] in $D = 4$ space–time is described in two–twistor space. The corresponding action is constructed from the following two–twistor one–form ($A = 1, ..., 4$ is the $SU(2,2)$ index)

$$\Theta^{(1)} = \sum_{i=1}^{2} \Theta_i^{(1)} = \sum_{i=1}^{2} \left(\bar{Z}^{Ai} dZ_{Ai} - d\bar{Z}^{Ai} Z_{Ai} \right) \tag{1}$$

with imposed suitable constraints.

In this report we show that the action determining the twistor formulation of the tensionful string is defined in two–twistor space via embedding of the canonical Liouville two–form in two–twistor space

$$\Theta^{(2)} = \Theta_1^{(1)} \wedge \Theta_2^{(1)} \,. \tag{2}$$

As we shall demonstrate, this twistor formulation is classically equivalent to the Hamiltonian formulation of $D = 4$ bosonic free string theory both with vectorial and tensorial string momenta.

2 Space–time formulations of the tensionful string

The tensionful string in flat Minkowski space is described by nonlinear Nambu–Goto action [1] [14, 15]

$$S = -T \int d^2\xi \sqrt{-\det(g_{mn})} \tag{3}$$

where $\xi^m = (\tau, \sigma)$ are world–sheet coordinates,

$$g_{mn} = \partial_m X^\mu \partial_n X_\mu \tag{4}$$

is the induced metric in string surface, T is the string tension.

Generally, the transition to twistorial formulation is realized via 'Hamiltonian' formulations. In case of the tensionful string (3) there are two Hamiltonian frameworks:

Formulation with vectorial momenta. The first order formulation of the tensionful string (3) is defined by the action [16]

$$S = \int d^2\xi \left[P_\mu^m \partial_m X^\mu + \tfrac{1}{2T} (-h)^{-1/2} h_{mn} P_\mu^m P^{\mu n} \right] \tag{5}$$

[1] The indices $m, n = 0, 1$ are vector world-sheet indices; $\mu, \nu = 0, 1, 2, 3$ is vector space–time ones.

used vectorial momenta variables $P_\mu^m(\xi)$. The kinetic part of the action (5) is described by the two–form

$$\tilde{\Theta}^{(2)} = P_\mu \wedge dX^\mu \tag{6}$$

where $P_\mu = P_\mu^m \epsilon_{mn} d\xi^n$, $dX^\mu = d\xi^m \partial_m X^\mu$ *i. e.* in the formulation (5) the pair (P_μ^0, P_μ^1) of generalized string momenta are represented by a one–form.

The equations of motion for world–sheet metric h_{mn}

$$P_\mu^m P^{n\mu} - \tfrac{1}{2} h^{mn} h_{kl} P_\mu^k P^{l\mu} = 0 \tag{7}$$

describe the Virasoro first class constraints.

Expressing P_μ^m in the action (5) by its equation of motion

$$P_\mu^m = -T(-h)^{1/2} h^{mn} \partial_n X_\mu \tag{8}$$

one obtains second-order action [17]

$$S = -\tfrac{T}{2} \int d^2\xi (-h)^{1/2} h^{mn} \partial_m X^\mu \partial_n X_\mu \,. \tag{9}$$

Formulation with tensorial momenta. Other formulation of the bosonic string (3) is the model with tensorial momenta. It is obtained from the Liouville two–form

$$\tilde{\tilde{\Theta}}^{(2)} = P_{\mu\nu} dX^\mu \wedge dX^\nu \,. \tag{10}$$

Such a model is directly related with the interpretation of strings as dynamical world sheets with the surface elements

$$dS^{\mu\nu} = dX^\mu \wedge dX^\nu = \partial_m X^\mu \partial_n X^\nu \epsilon^{mn} d^2\xi \,. \tag{11}$$

The string action with tensorial momenta is

$$S = \sqrt{2} \int d^2\xi \left[P_{\mu\nu}\, \partial_m X^\mu \partial_n X^\nu \epsilon^{mn} - \Lambda \left(P^{\mu\nu} P_{\mu\nu} + \tfrac{T^2}{4} \right) \right] . \tag{12}$$

Expressing $P_{\mu\nu}$ by its equation of motion, we get

$$P^{\mu\nu} = \tfrac{1}{2\Lambda} \Pi^{\mu\nu} \,, \qquad \Pi^{\mu\nu} \equiv \epsilon^{mn} \partial_m X^\mu \partial_n X^\nu \,. \tag{13}$$

It is important that the solution (13) satisfies the constraint $P^{\mu\nu}\tilde{P}_{\mu\nu} = 0$ as an identity where $\tilde{P}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho} P^{\lambda\rho}$. After substituting (13) in the action (12) we obtain the four–order action (see *e.g.* [18])

$$S = \tfrac{1}{2\sqrt{2}} \int d^2\xi \left[\Lambda^{-1} \Pi^{\mu\nu} \Pi_{\mu\nu} - \Lambda T^2 \right] . \tag{14}$$

Eliminating further Λ and using that

$$\Pi^{\mu\nu} \Pi_{\mu\nu} = 2 \det(g_{mn}) \tag{15}$$

we obtain the Nambu–Goto string action (3).

3 Tensionful string in mixed twistor-spacetime formulation

Mixed formulations with vectorial momenta. In order to obtain from the action (5) the mixed spinor–space-time action (18) we should eliminate the fourmomenta P^m_μ by means of the the string generalization of the Cartan–Penrose formula. On curved world sheet it has the form[2] [9]

$$P^m_{\alpha\dot\alpha} = e\,\tilde\lambda_{\dot\alpha}\rho^m\lambda_\alpha = ee^m_a\tilde\lambda^i_{\dot\alpha}(\rho^a)_i{}^j\lambda_{\alpha j}\,. \tag{16}$$

Then the second term in string action (5) takes the form

$$\tfrac{1}{2T}(-h)^{-1/2}h_{mn}P^m_\mu P^{n\mu} = \tfrac{1}{2T}\,e\,(\lambda^{\alpha i}\lambda_{\alpha i})(\tilde\lambda^j_{\dot\alpha}\tilde\lambda^{\dot\alpha}_j) \tag{17}$$

where we used $\mathrm{Tr}(\rho^m\rho^n) = 2h^{mn}$. Putting (16) and (17) in the action (5) we obtain the string action [19]

$$S = \int d^2\xi\, e\left[\tilde\lambda_{\dot\alpha}\rho^m\lambda_\alpha\,\partial_m X^{\dot\alpha\alpha} + \tfrac{1}{2T}\,(\lambda^{\alpha i}\lambda_{\alpha i})(\tilde\lambda^j_{\dot\alpha}\tilde\lambda^{\dot\alpha}_j)\right] \tag{18}$$

which provides the mixed space-time–twistor formulation of bosonic string. We stress that in the formulation [19] the twistor spinors $\lambda_{\alpha i}$ are not constrained. Further, the algebraic field equation (7) after substitution (16) is satisfied as an identity.

The action (18) invariants under the following local transformations:

$$\lambda'_{\alpha i} = e^{i(b+ic)}\lambda_{\alpha i}, \quad \bar\lambda'^{\,i}_{\dot\alpha} = e^{-i(b-ic)}\bar\lambda^i_{\dot\alpha}, \quad e'^{\,a}_m = e^{2c}e^a_m\,.$$

We can fix the real parameters b, c by imposing of the constraints

$$A \equiv \lambda^{\alpha i}\lambda_{\alpha i} - T = 0\,, \qquad \bar A \equiv \bar\lambda^i_{\dot\alpha}\bar\lambda^{\dot\alpha}_i - T = 0\,. \tag{19}$$

If we introduce the variables $v_{\alpha i} = \sqrt{\frac{2}{T}}\,\lambda_{\alpha i}$, $\bar v^i_{\dot\alpha} = \sqrt{\frac{2}{T}}\,\bar\lambda^i_{\dot\alpha}$ we get the orthonormality relations for the spinorial Lorentz harmonics [20].

If we impose the constraints (19) the model (18) can be rewritten in the following way

$$S = \int d^2\xi\left[ee^m_a\tilde\lambda^i_{\dot\alpha}(\rho^a)_i{}^j\lambda_{\alpha j}\,\partial_m X^{\dot\alpha\alpha} + \tfrac{T}{2}\,e + \Lambda A + \bar\Lambda\bar A\right] \tag{20}$$

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[2] $h_{mn} = e^a_m e_{na}$ is a world–sheet metric, e^a_m is the zweibein, $e^a_m e^m_b = \delta^a_b$, $e = \det(e^a_m) = \sqrt{-h}$. The indices $a, b = 0, 1$ are $d = 2$ flat indices. The indices $i, j = 1, 2$ are $d = 2$ Dirac spinor indices. We use bar for complex conjugate quantities, $\bar\lambda^i_{\dot\alpha} = \overline{(\lambda_{\alpha i})}$, and tilde for Dirac conjugated $d = 2$ spinors, $\tilde\lambda^i_{\dot\alpha} = \bar\lambda^j_{\dot\alpha}(\rho^0)_j{}^i$.

where the spinors λ, $\bar{\lambda}$ are constrained by the relations (19), which are imposed additionally in (20) by the Lagrange multipliers. It is easy to see that introducing the light cone notations on the world sheet for the zweibein $e_m^{++} = e_m^0 + e_m^1$, $e_m^{--} = e_m^0 - e_m^1$ and following Weyl representation for Dirac matrices in two dimensions we obtain harmonic string action [20, 21, 22].

Mixed formulations with tensorial momenta. The zweibein e_m^a can be expressed from the action (18) as follows

$$e_m^a = \tfrac{2T}{(\lambda\lambda)(\bar{\lambda}\bar{\lambda})}\ \tilde{\lambda}^i_{\dot{\alpha}}(\rho^a)_i{}^j\lambda_{\alpha j}\ \partial_m X^{\dot{\alpha}\alpha} \qquad (21)$$

Substitution of the relation (21) in the action (28) provides the following string action

$$S = \sqrt{2}\int d^2\xi\epsilon^{mn}\Big(P_{\alpha\beta}\,\partial_m X^{\dot{\gamma}\alpha}\partial_n X_{\dot{\gamma}}^{\beta} + \bar{P}_{\dot{\alpha}\dot{\beta}}\,\partial_m X^{\dot{\alpha}\gamma}\partial_n X_{\gamma}^{\dot{\beta}}\Big) \qquad (22)$$

where the composite second rank spinors

$$P_{\alpha\beta} = \tfrac{\sqrt{2}T}{(\lambda\lambda)}\,\lambda^1_{(\alpha}\lambda^2_{\beta)}\,, \qquad \bar{P}_{\dot{\alpha}\dot{\beta}} = \tfrac{\sqrt{2}T}{(\lambda\lambda)}\,\bar{\lambda}^1_{(\dot{\alpha}}\bar{\lambda}^2_{\dot{\beta})}\,. \qquad (23)$$

satisfy the constraints

$$P^{\alpha\beta}P_{\alpha\beta} = -\tfrac{T^2}{4}\,, \qquad \bar{P}^{\dot{\alpha}\dot{\beta}}\bar{P}_{\dot{\alpha}\dot{\beta}} = -\tfrac{T^2}{4}\,. \qquad (24)$$

Using fourvector notation $P_{\alpha\beta} = P_{\mu\nu}\sigma^{\mu\nu}_{\alpha\beta}$, $\bar{P}_{\dot{\alpha}\dot{\beta}} = -P_{\mu\nu}\sigma^{\mu\nu}_{\dot{\alpha}\dot{\beta}}$ the relations (24) take the form

$$P^{\mu\nu}P_{\mu\nu} = -\tfrac{T^2}{4}\,, \qquad P^{\mu\nu}\tilde{P}_{\mu\nu} = 0\,. \qquad (25)$$

Using additionally the conditions (19) the formulation (22) produces harmonic string action with tensorial string momenta (see also [23]).

4 Purely twistorial formulation

Let us introduce second half of twistor coordinates $\mu_i^{\dot{\alpha}}$, $\bar{\mu}^{\alpha i}$ by employing Penrose incidence relations generalized for string

$$\mu_i^{\dot{\alpha}} = X^{\dot{\alpha}\alpha}\lambda_{\alpha i}\,, \qquad \bar{\mu}^{\alpha i} = \bar{\lambda}^i_{\dot{\alpha}}X^{\dot{\alpha}\alpha}\,. \qquad (26)$$

Incidence relations (26) with real space–time string position $X^{\dot{\alpha}\alpha}$ imply that the twistor variables satisfy the constraints

$$V_i^j \equiv \lambda_{\alpha i}\bar{\mu}^{\alpha j} - \mu_i^{\dot{\alpha}}\bar{\lambda}^j_{\dot{\alpha}} \approx 0 \qquad (27)$$

which are antiHermitian ($(\overline{V_i{}^j}) = -V_j{}^i$).

Let us insert the relations (26) into (20). Using

$$P^m_{\alpha\dot{\alpha}}\partial_m X^{\dot{\alpha}\alpha} = \tfrac{1}{2} e e^m_a \left[\tilde{\lambda}_{\dot{\alpha}}\rho^a\partial_m\mu^{\dot{\alpha}} - \tilde{\mu}^\alpha\rho^a\partial_m\lambda_\alpha\right] + c.c.$$

we obtain the first order string action in twistor formulation

$$\begin{aligned} S \;&= \int d^2\xi \left\{ \tfrac{1}{2} e e^m_a \left[\tilde{\lambda}_{\dot{\alpha}}\rho^a\partial_m\mu^{\dot{\alpha}} - \tilde{\mu}^\alpha\rho^a\partial_m\lambda_\alpha + c.c.\right] + \right. \\ &\qquad\qquad \left. + \tfrac{T}{2}\, e + \Lambda_j{}^i V_i{}^j + \Lambda A + \bar{\Lambda}\bar{A} \right\} \end{aligned} \tag{28}$$

where $\Lambda_i{}^j = -(\overline{\Lambda_j{}^i})$, Λ, $\bar{\Lambda}$ are the Lagrange multipliers.

Introducing the string twistors

$$Z_{Ai} = (\lambda_{\alpha i}, \mu_i^{\dot{\alpha}}), \quad \bar{Z}^{Ai} = (\bar{\mu}^{\alpha i}, -\bar{\lambda}^i_{\dot{\alpha}}), \quad \tilde{Z}^{Ai} = \bar{Z}^{Aj}(\rho^0)_j{}^i,$$

the constraints (27) are rewritten as

$$V_i^j = Z_{Ai}\bar{Z}^{Aj} \approx 0\,. \tag{29}$$

Substituting the equations of motion for zweibein e^a_m

$$e^a_m = -\tfrac{1}{T}\left[\partial_m\tilde{Z}^A\rho^a Z_A - \tilde{Z}^A\rho^a\partial_m Z_A\right] \tag{30}$$

in the action (28) we obtain our basic twistorial string action [9]:

$$S = \int d^2\xi\,\mathcal{L}\,, \tag{31}$$

$$\begin{aligned} \mathcal{L} = \tfrac{1}{4T}\epsilon^{mn}\epsilon_{ab}\left[\partial_m\tilde{Z}^B\rho^a Z_B - \tilde{Z}^B\rho^a\partial_m Z_B\right]\left[\partial_n\tilde{Z}^A\rho^b Z_A - \tilde{Z}^A\rho^b\partial_n Z_A\right] + \\ +\Lambda_j{}^i V_i^j + \Lambda A + \bar{\Lambda}\bar{A}\,. \end{aligned} \tag{32}$$

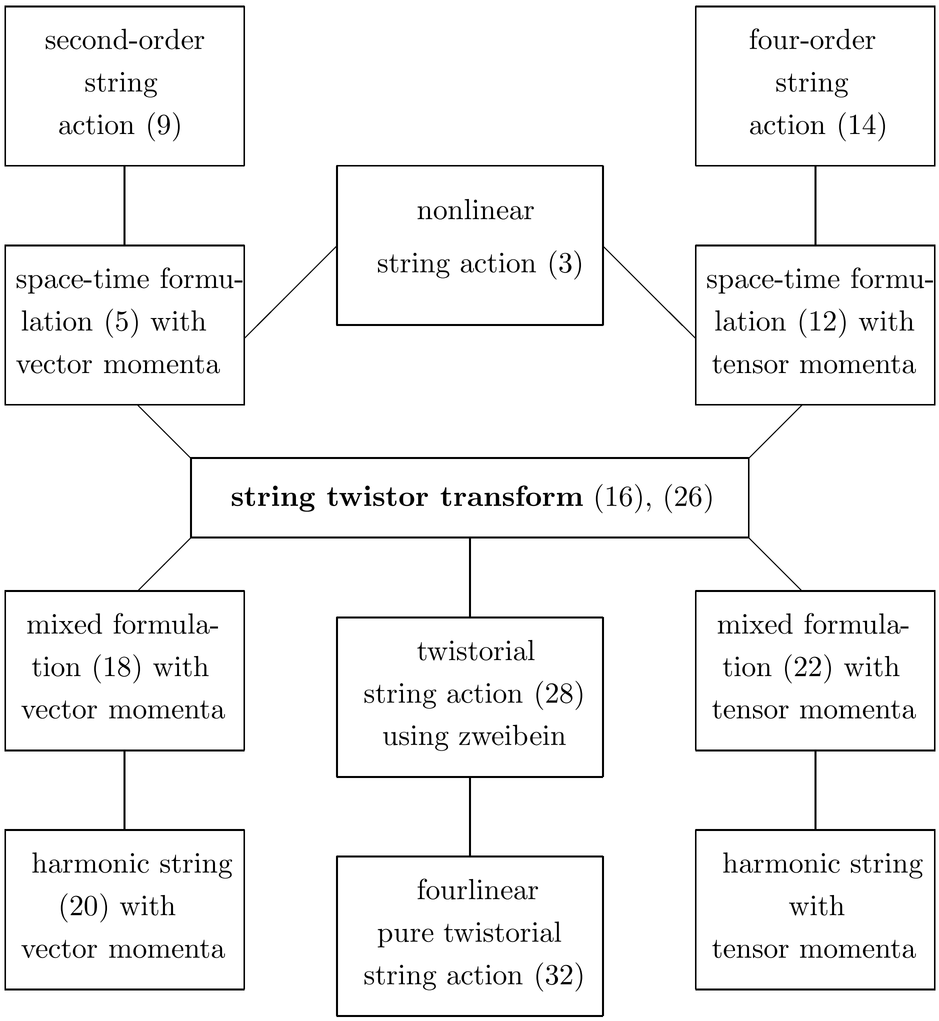
Using explicit form of $D=2$ Dirac matrices we obtain that the first term in the Lagrangian (32) equals to

$$\tfrac{1}{T}\epsilon^{mn}\left[\partial_m\bar{Z}^{A1}Z_{A1} - \bar{Z}^{A1}\partial_m Z_{A1}\right]\left[\partial_n\bar{Z}^{B2}Z_{B2} - \bar{Z}^{B2}\partial_n Z_{B2}\right] \tag{33}$$

i.e. the action (32) is induced on the world–sheet by the canonical 2–form (2) with supplemented constraints (19) and (27).

5 Conclusions

In this report we presented the classical master twistor formulation of tensionful string (31) and its links with various bosonic string models. These links can be represented by the following diagram:



There are several ways to extend the studies presented briefly in this report which are now under consideration:

i) One can consider the Hamiltonian formulation of the twistorial string action (32), with first and second class constraints. One should observe a convenient property of the fourlinear action (32) – it is linear in time derivative, in analogy to the momentum formulations of the superparticle models. The aim of our studies is to obtain the Gupta-Bleuler type of quantization of the model (32).

ii) One can show that the PB of the constraints (29) describe $U(2)$ algebra. Following the discussion of massive spinning particle model in 2-twistor space [12]–[13]. one can interpret the four $V_i^{\,j}$ generators as describing local density of spin and electric charge of the string.

iii) The presented links between various bosonic string models can be extended supersymmetrically in two-supertwistor space, by applying already developed supersymmetrization techniques [19, 20].

iv) In the approach of Witten and Berkovits [7, 8] the twistor string model is endowed with single (super)twistor target space, and the correspondence with Penrose theory is obtained only on the level of quantized twistorial string. Our twistorial model is classically equivalent to the standard bosonic string model. If the quantization of our model can be achieved, it will be possible to relate our and Witten's approach, which we expect will strictly overlap only for very restricted class of string modes.

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